Successive linearization in model-based control for thermal management systems

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Abstract—We propose a successive linearization algorithm for dynamical model-based control of aircraft thermal management systems. The increasing electrical and thermal loads on-board of air vehicles, driven largely by the trend of electrification, present a significant challenge to the design and operation of thermal management systems. Components of thermal management systems are typically designed offline using a steady-state analysis given mission profiles. Dynamical system models can be used online during the operation of the aircraft as model-based control to reject disturbances in real-time. Real-time optimal control for aircraft thermal management system is challenging for multiple reasons. First, the problem is not convex due to the thermal system dynamics. Second, finding a good operating point (or trajectory) for linearization is not trivial, and the selected linearization point might not capture the actual system dynamics. Successive linearization is an algorithm to find a good linearization trajectory in a systematic manner by first convexifying the nonlinear constraints, and iteratively converging to trajectories that ensure that the nonlinear dynamics are satisfied. In particular, the algorithm converts the original problem into a sequence of second-order cone programs that can be solved efficiently and reliably in real-time using standard off-the-shelf solvers.

I. INTRODUCTION

Driven by the needs for higher efficiency and lower audible noise, increasing effort is made towards moving to a more electrical aircraft. This paradigm-shift of replacing the traditional pneumatic or hydraulic system with an electric one is accompanied with new challenges. The electric power demand is much higher than that of a conventional approach. For example, the total electrical power generation on Boeing 787 is above 1MW while a corresponding conventional aircraft uses less than one fourth of the power [1]. Significantly higher electrical power demand challenges the thermal management systems. In addition, the use of new composite materials of high thermal resistance in the aircraft skin can further increase these challenges for aircraft operating at high Mach numbers because high speed contributes to skin-friction heating [2].

There have been some approaches recently in the control community to address these thermal challenges. Reference [3] considers improved topologies for thermal management, [4] tackles the thermal management problem through mission planning. There has also been work on model predictive framework with hierarchical control [5]–[8]. These works share the common thread of a transient treatment of thermal management, increasing the fidelity of traditional steady-state analyses [9].

While ram air is the main source of thermal dissipation, some aircraft use fuel as a heat sink. Fuel thermal endurance is defined as the time for the fuel temperature in any part of a fuel-cooled aircraft to reach to a certain limit [10]. In this paper we study fuel thermal management system through the control of valves, pumps, heat exchangers, in the fuel-loop system. Controlling the mass flow in the fuel-loop from the tank to the engine is important as actions performed early in the flight can significantly affect the aircraft thermal endurance. First, we model the fuel thermal system using graph-based dynamical system modeling where each thermal component is represented as a vertex in this graph and a link connecting two vertices in the graph corresponds to a power flow between components. The graph-based framework has been investigated in [6], [11], [12]. Using conservation of energy and conservation of mass, the dynamical system equations are derived as function of the system variables (temperature of components) and control variables (mass flow rate of fuel).

The ability to increase the endurance of fuel thermal management system brings along stringent challenges. The nonlinear dynamics of the energy dissipation equations coupled with various thermal disturbances from avionics, actuators, engine components, and weapon systems are major sources that challenges real-time optimal control of thermal management system. As a result, maximizing endurance depends on the full exploitation of performance envelope i.e., controlling the temperature of components relies on the ability to generate control trajectories on the limits of physically feasible scenarios. Recently, model predictive control (MPC) has been proposed to model the thermal system with temperature constraints [6], [12]. Global optimization techniques have been used to address the nonlinearity in the dynamics which makes the approach numerically intractable for fast real-time implementations as required for thermal management applications. Linearizing around the steady-state in the MPC [10] gives dynamics that are not consistent with the nonlinear dynamics which makes the controller ineffective. We address this issue in this paper using successive convexification [13], [14], which deals with this problem by first convexifying the nonlinear constraints, and iteratively converging to trajectories that ensure that the nonlinear dynamics are satisfied. In particular, the contribution of this paper is to convert the original problem into a sequence of convex programs, i.e., successive linearization, that can be solved efficiently and reliably in real-time using any standard convex optimization solver. An important advantage of the successive convexification for thermal management systems...
is that the algorithm can be initiated by starting from simple, dynamically inconsistent linearization points (or trajectories).

II. PROBLEM FORMULATION

A. Graph Preliminaries

The thermal management system to be controlled can be represented by a directed graph $G = (V, \mathcal{E})$ where $V = \{v_1, \ldots, v_n\}$ is the set of vertices corresponding to thermal components that store energy, and $\mathcal{E} \subseteq V \times V$ is the set of links corresponding to power flow between components, i.e. $(i, j) \in \mathcal{E}$ if power flows from thermal component $i$ to thermal component $j$. We define the set of vertices for an incoming power flow into component $i$ to be $V_i^m := \{j \in V; (j, i) \in \mathcal{E}\}$, and the set of vertices for an outgoing power flow from $i$ to be $V_i^q := \{j \in V; (i, j) \in \mathcal{E}\}$. Let $P_{ij}$ be the power flow through link $(i, j)$. If $(i, j) \notin \mathcal{E}$, then $P_{ij} = 0$.

B. System Description and Modeling

A thermal system can be modeled as a network of connected thermal components. Fig. 1 models a given thermal component in the system. The dynamics of each thermal component (and the overall system) can be derived from the law of conservation of energy and the law of conservation of mass. The law of conservation of energy imposes the constraint that the rate of change of energy in a component $E_i$ is the sum of power flows out of the component minus the sum of power flow into the component:

$$E_i = \dot{q}_i + \sum_{j \in V_i^m} P_{ji} - \sum_{k \in V_i^m} P_{ik}, \quad \text{(1)}$$

where $\dot{q}_i$ is the external input/output heat flow at component $i$ (in kW).

The law of conservation of mass imposes the constraint that the sum of the input mass flow to a thermal component $i$ is equal to the sum of output mass flow out of that component:

$$\sum_{j \in V_i^m} \dot{m}_{ji} = \dot{m}_{i,\text{out}} + \sum_{k \in V_i^m} \dot{m}_{ik}, \quad \text{(2)}$$

where $\dot{m}_{ij}$ is the mass flow (in kg/s) through link $(i, j)$, and $\dot{m}_{i,\text{out}}$ is the mass flow out of the system (e.g., fuel flowing to the engine).

C. System Dynamics

The focus of this paper is to control a fuel thermal management system (FTMS) where the fuel in a loop circulates from the fuel tank to the engine, and absorbs heat from different avionics along the way. In some aircraft, the FTMS provides cooling to many components including the generator, engine oil, oil pumps, fuel pumps, and advanced electrical equipment. We will first start by deriving the dynamical equations that governs the change in temperature in components as function of the mass flow rates of the fuel. To study the system dynamics of FTMS system, we consider one fuel tank denoted by $F$, connected to a number of other thermal components forming a directed graph $G$ where $F \in V$. Having the tank as a separate component is necessary because of the mass depletion dynamics of the fuel, i.e., the mass of the fuel is decreasing and should be considered in the dynamics, and the output mass flow rate from the tank is controlled by a pump. The extension to having multiple fuel tanks (see for example [10]) is straightforward. The thermal energy at the tank is given by

$$E_F = M_F c_p T_F \quad \text{(3)}$$

where $M_F$ is the mass of fuel, $c_p$ is the specific heat, and $T_F$ is the temperature of the fuel. Both, $M_F$ and $T_F$ are state variables, and differentiating the above equation with respect to time gives:

$$\dot{E}_F = M_F c_p T'_F + M_F c_p \dot{T}_F. \quad \text{(4)}$$

From conservation of mass in the system, the rate of change of mass in the fuel tank $M_F$ is equal to negative of the mass flow rate out of the system $\sum_{i \in V} \dot{m}_{i,\text{out}}$, i.e.,

$$\dot{M}_F = -\sum_{i \in V} \dot{m}_{i,\text{out}}. \quad \text{(5)}$$

As a result, the conservation of mass flow equation at the tank is:

$$\sum_{j \in V_F^m} \dot{m}_{jF} = \dot{m}_{F,\text{out}} + \sum_{k \in V_F^m} \dot{m}_{Fk}, \quad \text{(6)}$$

The thermal energy at the other components of the system is modeled as

$$E_i = C_i T_i \quad \text{for } i \in V \setminus \{F\} \quad \text{(7)}$$

where $C_i$ is the thermal capacitance (in kJ/K) of the component, assumed constant.

The power flow through a link $(i, j)$ in the directed graph $G$ can be given as follows:

$$P_{ij} = \dot{m}_{ij} c_p T_i \quad \text{(8)}$$

where $\dot{m}_{ij}$ is the mass flow rate through the link $(i, j)$, and $T_i$ is the temperature of the “head” vertex of the link. Substituting equations (7) and (8) in the conservation of energy equation (1), and using the conservation of mass property in (2), $\sum_{j \in V_i^m} \dot{m}_{ji} = \dot{m}_{i,\text{out}} + \sum_{k \in V_i^m} \dot{m}_{ik}$, we
get for all \( i \in V \setminus \{ F \} \):
\[
\dot{T}_i = \frac{\dot{q}_i + \sum_{j \in V_i^p} \dot{m}_{ij} c_p T_j - \sum_{k \in V_{i,ou}} \dot{m}_{ik} c_p T_i}{C_i} = \frac{\dot{q}_i + \sum_{j \in V_i^p} \dot{m}_{ij} c_p T_j - (\sum_{j \in V_i} \dot{m}_{ij} - \dot{m}_{i,ou}) c_p T_i}{C_i} = \frac{\dot{q}_i + \dot{m}_{i,ou} c_p T_i + \sum_{j \in V_i^p} \dot{m}_{ij} c_p (T_j - T_i)}{C_i}.
\]

(9)

The dynamics of the tank \( F \) is given by substituting (4) and (8) in the conservation of energy equation (1), and using the conservation of mass property in (6). Assuming in addition that \( \dot{q}_F = 0 \), we get
\[
M_F \dot{T}_F + M_F T_F = (\dot{m}_{F,ou} + \dot{M}_F) T_F + \sum_{j \in V_F^p} \dot{m}_{jF}(T_j - T_F),
\]
and therefore,
\[
\dot{T}_F = \frac{\dot{m}_{F,ou} T_F + \sum_{j \in V_F^p} \dot{m}_{jF}(T_j - T_F)}{M_F}.
\]

(10)

III. MODEL-BASED CONTROL

For model-based control, we put the system in state space form. We define the state vector \( x(t) \in \mathbb{R}^{1|V|+1} \) whose elements are given by the set:
\[
\{ [T_i(t)]_{i \in V}, F \}.
\]

(11)
The control in FTMS system is characterized by controlling the mass flow on some of the links \( \dot{m}_{ij} \) (through controlling the pump and subset of the valves). Let \( U \) be the set of possible control variables. The control \( u(t) \in \mathbb{R}^{|U|} \) where \(|U|\) is the number of control variables in \( U \) to be:
\[
u(t) = [u_i(t)]_{i \in U},
\]
and the dynamics
\[
\dot{x}(t) = f(x(t), u(t))
\]
are given by equations (5), (9), and (10).

A standard method to deal with the nonlinearities in the dynamics is to linearize the system around an operating point \((\bar{x}(t), \bar{u}(t))\) by using a first-order Taylor series approximation of \( f(x(t), u(t)) \):
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) + z(t)
\]
where
\[
A(t) := \frac{\partial f(x,u)}{\partial x}|_{(x,t), (u,t)}, \quad B(t) := \frac{\partial f(x,u)}{\partial u}|_{(x,t), (u,t)}, \quad \text{and} \quad z(t) := f(\bar{x}(t), \bar{u}(t)) - A(t)\bar{x}(t) - B(t)\bar{u}(t).
\]

To add constraints to the system and solve the optimal control problem using digital controllers, the continuous-time dynamics can be discretized and the problem is then written as finite-dimensional parameter optimization. Let \( T_s \) be the sample period, then we define the discrete time instants as \( t_k = kT_s, k = 1, 2, ... \). For notation convenience, we define \( x(k) := x(t_k) \) and \( u(k) := u(t_k) \), and let \( u := \{u(0), u(1), \ldots, u(k-1)\} \) and \( x := \{x(0), x(1), \ldots, x(k)\} \) for a finite horizon \( k \). We use Euler forward discretization where \( \dot{x} \approx \frac{x(k+1) - x(k)}{T_s} \) in the simulations, but a more general discretization based on zero-order hold (ZOH) can also be used. The discrete-time state-space system can be written as follows
\[
x(k + 1) = A_k x(k) + B_k u(k) + z_k
\]
where \( A_k = T_s A(t_k) + I, B_k = T_s B(t_k), \) and \( z_k = T_s z(t_k) \).

The objective of the control system of the FTMS is for the temperature after the heat is rejected to track a reference, i.e., \( T_i \approx T_i,ref \) for \( i \in S \) where \( S \subseteq V \). These references are chosen by a higher level control system in such a way that tracking these values would maximize the endurance of the FTMS [10].

The optimal control problem of the FTMS system can be written as follows:
\[
\begin{align*}
\text{minimize}_{u, x} & \quad g(x, u) \\
\text{subject to} & \quad x(k + 1) = A_k x(k) + B_k u(k) + z_k, \quad \text{for all } k \\
& \quad x(0) \in \mathcal{X}_0, x \in \mathcal{X}, u \in U
\end{align*}
\]

where \( g(x, u) \) is the cost function, \( \mathcal{X}_0 \) is the initial conditions set, \( \mathcal{X} \) is the constraint set of the state and \( U \) is the constraint set of control.

To track a reference, we add auxiliary variables \( \delta_i \in \mathbb{R}^k \) and \( \delta_i \in \mathbb{R}^k \) for \( i \in S \) such that the following soft constraints hold:
\[
T_i - T_i,ref \leq \delta_i, \quad \delta_i \geq 0 \quad \text{(16)}
\]
\[
T_i,ref - T_i \leq \delta_i, \quad \delta_i \geq 1 \quad \text{(17)}
\]
where \( \delta_i \) is penalized in the cost function to ensure that \( T_i \) is below \( T_i,ref \) at all times, and \( \delta_i \) is penalized to ensure that the temperature is within one degree Kelvin from the reference. Note that it is desirable that the penalty for \( \delta_i \) be higher than that of \( \delta_i \) because the thermal management system prefers the temperatures to be below the reference than above it. Along with tracking the references, the system would also have hard constraints on the temperature in all components:
\[
T_i \leq T_{\text{max}} \quad \text{for } i \in V. \quad \text{(18)}
\]

In addition to the state constraints, the system has control constraints. In general, control constraints are defined as convex polytopic regions given as follows,
\[
F_k u_i(k) \leq b_k \quad \text{for } k = 1, \ldots, k - 1 \quad \text{(19)}
\]
where \( F_k \) and \( b_k \) are given. For the FTMS system, constraints on the rate of change of the control are added, i.e., for \( i \in U \) we have
\[
|u_i(k + 1) - u_i(k)| \leq u_{\text{maxchange}}, \quad \text{for } k = 1, \ldots, k - 1, \quad \text{(20)}
\]
where \( u_{\text{maxchange}} \in \mathbb{R} \). In addition, the optimization should include constraints that ensure the total mass flow out of the tank is more than the mass flow exiting to the engine.

Let \( \delta = \{\delta_i, \delta_i\} \in S \), the cost function of the optimization
The controller from the linearized model-based optimization (15) produces nonlinear trajectories that drift away from the linear dynamics. In this example, $T_{i,\text{ref}} = 315$ and the figure shows that the linear system satisfies all constraints and requirements, but the nonlinear (actual) system violates the constraints.

Problem can be given as follows:

$$g(x, u, \delta) = \sum_{i \in S} (\alpha_1 ||\delta_i||_2 + \alpha_2 ||\delta_i||_2) + \sum_{j \in U} \alpha_3 ||u_j||_2,$$

where $\alpha$’s are the costs associated with penalizing the L-2 norm of auxiliary states or control. We have $\alpha_1 \gg \alpha_2$ because the system incurs higher cost when the temperature exceeds the tracking reference temperature than being below the reference, $\alpha_3$ is the cost of the control effort.

A. Successive Linearization

The model-based control is effective when the nonlinear dynamics $\dot{x} = f(x, u)$ is well characterized by the linearized system equations (13). Due to the bilinearities in the FTMS dynamics, it turns out that the operating point for linearization is an important factor for the linearized system to be a good representation of the nonlinear dynamics. Fig. 2 shows that the linear model from the convex optimization might not be a good characterization of the nonlinear dynamics for a given linearization trajectory around which the nonlinear dynamics are linearized. Finding a good linearization point is not trivial, and linearizing around steady-state operating points is not robust to heat disturbances. This information that can be available for the control system in real-time to tune the controllers. Note that adding state constraints in the model-based control can render the linearized system infeasible. In other words, there might not be a control input that makes the system satisfy the constraints even if the nonlinear dynamical equations have a feasible solution, the so called artificial infeasibility. Previous work in model-based control of FTMS has mainly used global optimization methods [6], but suffers from high computational complexity and do not scale for larger graphs or longer prediction time horizon.

Successive linearization is an iterative process where the input is the nonlinear model $\dot{x} = f(x, u)$ and the output is the control $u$ that minimizes a cost function while the system satisfy state constraints. Successive linearizations deal with the challenges presented by (13) in the following ways. First the dynamics are relaxed by introducing a new variable $v(t)$ to the dynamics (13):

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + z(t) + v(t). \quad (21)$$

Adding $v(t)$ prevents artificial infeasibility due to the linearization [13]. Let $v_k := v(k)$ be the discretized vector, and denote $v := [v_k]_{k=1}^{k-1}$. To get the feasible solution when it exists, we penalize $||v||_2$ in the cost function by a high weight, $\alpha_4 \gg 0$. Note that if all state constraints are soft constraints, then feasibility is not a problem because there will always be a feasible solution for the linearized system and for $\alpha_4$ big enough we have $||v|| \approx 0$ for all iterations in the successive iterations. But this is an important step for more complex systems with hard state constraints as thermal management systems having strict requirements on the maximum temperature allowed.

Another important aspect in successive linearization is the convergence of the iterative process. Techniques based on trust region methods have been used in the literature [13], [14] whose purpose is to avoid the system to drift away from the operating point of linearization $(\bar{x}, \bar{u})$. This is achieved by introducing new variables $\mu \in \mathbb{R}^{k-1}$, $\Delta x_k := x(k) - \bar{x}(k)$, $\Delta u_k := u(k) - \bar{u}(k)$, and adding the following second-order cone (SOC) constraints for $k = 1, \ldots, k-1$:

$$||\Delta x_k||_2^2 + ||\Delta u_k||_2^2 \leq \mu_k, \quad \mu_k \geq \mu_{\text{min}} \quad (22)$$

where $\mu_{\text{min}}$ is given and represents how far the system is allowed to deviate from the nominal trajectory. The variable $||\mu||$ is penalized in the cost function by a weight $\alpha_5$. For example, if $\mu_{\text{min}} = 0$ and $\alpha_4$ is big enough, then the system would operate very close to the nominal trajectory $(\bar{x}(k), \bar{u}(k), k = 1, 2, \ldots)$. The overall optimization in a successive linearization iteration is the following:

\[
\begin{align*}
\text{minimize} & \quad g(x, u, \delta) + \alpha_4 ||v||_1^2 + \alpha_5 ||\mu||_1^2 \\
\text{subject to} & \quad x(k+1) = A_k x(k) + B_k u(k) + z_k + v_k, \quad \forall k \\
& \quad T_{i} - T_{i,\text{ref}} \leq \bar{\delta_i}, \quad \bar{\delta_i} \geq 0, \\
& \quad T_{i,\text{ref}} - T_{i} \leq \bar{\delta_i}, \quad \bar{\delta_i} \geq 1, \\
& \quad F_k u_j(k) \leq b_k, \quad \forall k \\
& \quad ||\Delta x_k||_2^2 + ||\Delta u_k||_2^2 \leq \mu_k, \quad \mu_k \geq \mu_{\text{min}}, \quad \forall k \\
& \quad T_i \leq T_{\text{max}} \\
& \quad x(0) = x_0.
\end{align*}
\]

(23)

The standard approach for the successive linearization is given in Fig. 3, see [13]. Starting from initial operating point $(\bar{x}^0, \bar{u}^0)$, the successive iteration $i$ uses the nonlinear discrete-time dynamics denoted by $f_d(x_k, u_k)$ and linearize around the operating point. The model-based control is then solved using the convex program (23). The output of the optimization is $(\bar{x}^{i,\text{opt}}, \bar{u}^{i,\text{opt}})$, where $\bar{u}^{i,\text{opt}}$ is used in the nonlinear dynamics to generate trajectories $\bar{x}^{i+1}$ for the
following successive linearization iteration. The process is repeated until the nonlinear trajectories $\tilde{x}_{i+1}$ are within a predefined tolerance $\epsilon$ of the optimizer output $x^{\text{opt}}$. The intuition behind the overall process is that upon convergence, the designed controller would optimize the objective function given the nonlinear dynamics, i.e., it will converge to a local optimum of the nonlinear global optimization. That is particularly useful for problems where the objective function is convex, and the only nonlinearities are presented in the dynamics, i.e., the sets $X$ and $U$ in (15) are convex, as in the FTMS example considered.

Theoretic guarantees on the convergence of the successive iterative procedure are usually established by contracting the trust region radius $\mu_{\text{min}}$ with every iteration, see [14] for more details. In the case of our FTMS example, the convergence turns out to be very fast, within few number of iterations even without contracting the trust region radius and using a fixed bound $\mu_{\text{min}}$ for all iterations, this is mainly because we have mild nonlinearities, i.e., bilinear terms.

IV. SUCCESSIVE LINEARIZATION APPLIED TO FTMS EXAMPLE

A simplified system for the FTMS can be shown in Fig. 4, see [6], [10], [12]. Starting from the tank, the fuel flows through a loop by adjusting a variable-speed pump, after which the fuel splits into two heating loops. We divide the cooling components into two categories: low frequency with slow changes in heat flow rate $\dot{q}_L$, and high frequency with fast changes in heat flow rate $\dot{q}_H$. The heat added to the system is considered as exogenous inputs (i.e., disturbances) from the control perspective of the fuel thermal management system. The heating loop (L) corresponds to heat rejected from low frequency components where $\dot{q}_L$ is added to the system. Similarly, the heating loop (H) corresponds to heat rejected from high frequency components where $\dot{q}_H$ is added to the system. $\dot{m}$ is the mass flow rate of the fuel (in kg/s). Downstream from the heating loops, part of the fuel flow goes to the engine and the remaining part passes through a ram air heat exchanger to reject heat outside the aircraft before it goes back into the tank. The valve opening ratio $b \in [0, 1]$ provides how much portion of the fuel to be cooled, if the valve is fully open, i.e., $b = 1$ then all the fuel passes through the ram air and the system provides higher cooling capability. Note that passing all the fuel through the ram air exchanger introduces thrust penalty into the system, i.e., cooling through the ram air incurs a cost on the system.

As a result of the mass flow to the engine, the total mass $M_F$ decreases as a function of time and the thermal energy at the tank is a function of both its temperature $T_F$ and its mass $M_F$. We will derive the dynamic equations that govern this change in the next subsection, but first let us give the graph model for the above system. The graph model in given in Fig. 5, where each vertex in the figure corresponds to a thermal component in FTMS system.

A. System Dynamics

The dynamics of the FTMS system is based on the conservation of energy equation (1). The state variables of the system are the temperature of every thermal component, $T_F, T_L, T_H, T_J, T_R$, and the mass of the fuel in the tank, $M_F$. Note that since the aircraft is consuming fuel, $M$ is a decreasing function of time and is directly related to the mass flow of fuel to the engine $\dot{m}_E$. The control actuators are the pump, and valves, therefore the control variables in this system are the mass flow rates $\dot{m}_L$ and $\dot{m}_H$, and valve
The heat added to the system \( \dot{q}_L, \dot{q}_H, \) and engine fuel mass flow \( \dot{m}_E \) are exogenous processes and are considered as disturbances. The heat rejected to ram air \( \dot{q}_R \) is fixed.

The dynamics of the tank can be derived as in (5), (9), and (10). The overall dynamical equations are given by:

\[
\begin{align*}
\dot{T}_F &= \frac{(\dot{m}_L + \dot{m}_H - \dot{m}_E)(T_R - T_F)}{M_F} \quad (24) \\
\dot{T}_L &= \dot{q}_L + \dot{m}_L c_p (T_F - T_L) / C_L \quad (25) \\
\dot{T}_H &= \dot{q}_H + \dot{m}_H c_p (T_F - T_H) / C_H \quad (26) \\
\dot{T}_J &= c_p(\dot{m}_L T_L + \dot{m}_H T_H - (\dot{m}_L + \dot{m}_H) T_J) / C_J \quad (27) \\
\dot{T}_R &= -b \dot{q}_H + (\dot{m}_L + \dot{m}_H - \dot{m}_E)c_p(T_J - T_R) / C_R \quad (28) \\
\dot{M}_F &= -\dot{m}_E. \quad (29)
\end{align*}
\]

Equations (24)-(29) are bilinear in \( \dot{m} \) and \( T \) where two of the control variables, \( \dot{m}_L \) and \( \dot{m}_H \), are presented as a multiplicative factor with the state variables, \( T \), the temperature of components. Also, we have to deal with the challenge that the mass depletion in the tank is presented in the denominator of components. Also, we have to deal with the challenge that the temperature after the heat is rejected to track a reference, \( T \), the control variables, \( u \).

The objective of the control system of the FTMS is for the temperature after the heat is rejected to track a reference, \( T \), the control variables, \( u \).

We define the state vector \( x(t) \in \mathbb{R}^6 \) to be:

\[
x(t) = [T_F \ T_L \ T_H \ T_J \ T_R \ M_F ]^T, \quad (30)
\]

and the control \( u(t) \in \mathbb{R}^3 \) to be:

\[
u(t) = [\dot{m}_L \ \dot{m}_H \ b]^T, \quad (31)
\]

and the dynamics \( \dot{x}(t) = f(x(t), u(t)) \) is given by equations (24)-(29).

The objective of the control system of the FTMS is for the temperature after the heat is rejected to track a reference, \( i.e., T_L \approx T_{L,ref} \) and \( T_H \approx T_{H,ref} \).

### B. FTMS Model-Based Control

The total mass flow out of the tank cannot be below the mass flow exiting to the engine, \( i.e., \)

\[
\dot{m}_E \leq \dot{m}_L + \dot{m}_H \leq \dot{m}_{\text{max}} \quad (32)
\]

where \( \dot{m}_{\text{max}} \in \mathbb{R}^k \) is the maximum mass flow rate the pump is allowed to circulate. We also add constraints on how fast the control can be changed, \( i.e., \) for \( i \in \{H, L\} \) we have

\[
|\dot{m}_i(k+1) - \dot{m}_i(k)| \leq \dot{m}_{\text{maxchange}}, \quad \text{for } k = 1, \ldots, \hat{k} - 1, \quad (33)
\]

where \( \dot{m}_{\text{maxchange}} \in \mathbb{R} \).

Let \( \delta = \{\delta_H, \delta_L, \delta_H, \delta_L\} \), the cost function of the optimization problem can be given as follows

\[
g(x, u, \delta) = \sum_{i \in \{H, L\}} (\alpha_1 ||\delta_i||_2 + \alpha_2 ||\delta_i||_2 + \alpha_3 ||\dot{m}_i||_2),
\]

where \( \alpha_j \)'s are the costs associated with penalizing the L-2 norm of auxiliary states or control. We have \( \alpha_1 \gg \alpha_2 \) because the system incurs higher cost when the temperature exceeds the tracking reference temperature than being below the reference. Here, \( \alpha_3 \) is the coefficient for the control effort.

The convex optimization for the FTMS system in Fig. 5 is given from equation (23) by the following optimal control problem

\[
\text{minimize} \quad g(x, u, \delta) + \alpha_4 ||v|| + \alpha_5 ||\mu||
\]

subject to

\[
x(k + 1) = A_k x(k) + B_k u(k) + z_k + v_k, \quad \forall k
\]

for \( i \in \{L, H\} \):

\[
T_i - T_{i,ref} \leq \delta_i, \quad \delta_i \geq 0,
\]

\[
T_{i,ref} - T_i \leq \delta_i, \quad \delta_i \geq 1,
\]

\[
|\dot{m}_i(k+1) - \dot{m}_i(k)| \leq \dot{m}_{\text{maxchange}}, \quad \forall k
\]

\[
\dot{m}_E \leq \dot{m}_L + \dot{m}_H \leq \dot{m}_{\text{max}},
\]

\[
||\Delta x_k||_2 + ||\Delta u_k||_2 \leq \mu_k, \quad \mu_k \geq \mu_{min}, \quad \forall k
\]

\[
T_j \leq T_{\text{max}}, \quad \forall j \in V
\]

where \( M_{F,0} \) is the initial mass in the tank, and \( T_{j,0} \) is the initial temperature of component \( j \).

### C. FTMS Successive Linearization Convergence

Fig. 6 shows that the temperature \( T_H \) given as the output of the convex program (23) converges to a trajectory close to the output of the nonlinear dynamics within couple of iterations.

The advantage of having successive iterations in this case is that if only a single linearization is performed, then the convex optimization would give costs that are not a good indication of the real costs incurred following the nonlinear dynamics. In particular, the \( T_H \) from linear model in the first iteration shows that the temperature gets below the reference temperature 315 Kelvin within approximately 30 seconds and remains below the reference for all the remaining time horizon, while applying the optimal control on the nonlinear system reveals that the temperature exceeds the reference for most of the considered time horizon. Therefore, one iteration linearization (i.e., the standard linearization technique) would not capture the characteristics of the real system to be controlled. In the second iteration, however, the output of the optimization is close to the nonlinear dynamics indicating that the cost given by the convex optimization is a good...
approximation of what will be the cost of the real system.

Table I shows the error $||x^{i+1} - x_{i,\text{opt}}^i||_2$ during the convergence process when applying the successive linearization procedure to the FTMS problem. Within 5 iterations, the linear dynamics trajectories converge to the nonlinear dynamics trajectory, thus providing a local minimum of the original non-convex problem.

### D. Simulation Results on FTMS Example

We apply in this section the successive linearization procedure to the FTMS example described in Section IV. The specific heat $c_p$ and the thermal capacitance of components $C$ are considered constant, where $c_p = 4.181$ kJ/(kg.K) and $C_i = 209$ kJ/K for $i \in V \setminus \{F\}$. The reference temperatures that the system should track are the temperatures of the thermal components in the set $S = \{L, H\}$ given by $T_{H,\text{ref}} = 315$ K and $T_{L,\text{ref}} = 313$ K. The system starts with initial values $T_H(0) = 317$ K and $T_L(0) = 313$ K and the control is the mass flow rates $\dot{m}_L$ and $\dot{m}_H$ that should be designed by the controller to track the reference temperatures. Note that in this simulation we set $b = 0$ as fixed value, i.e., there is no cooling with ram air, and that would make the control effort to track the reference

These values correspond to water circulating in the FTMS loop as in [12].

| Iterations | $||T_{i+1}^L - T_{L,\text{opt}}^i||$ | $||T_{i+1}^H - T_{H,\text{opt}}^i||$ | $||T_i^J - T_{J,\text{opt}}^i||$ | $||T_{i+1}^F - T_{F,\text{opt}}^i||$ |
|------------|----------------------------------|----------------------------------|------------------------|----------------------------------|
| Iteration 1 | 28.2668                          | 18.7014                          | 2.4193                 | 137.1722                         |
| Iteration 2 | 0.3343                           | 0.7052                           | 0.0369                 | 0.0225                           |
| Iteration 3 | 0.0407                           | 0.0003                           | 0.0036                 | 0.0032                           |
| Iteration 4 | 0.0040                           | 0.0003                           | 0.0003                 | 0.0003                           |
| Iteration 5 | 0.0003                           | 0.0003                           | 0.0003                 | 0.0003                           |

TABLE I: Successive linearization convergence on the FTMS example.

Fig. 7: The profiles of the heat loads added to the system and the mass flow rate to the engine.

Fig. 8: Controller output satisfies constraints enforced by the optimization problem.

We apply in this section the successive linearization procedure to the FTMS example described in Section IV. The specific heat $c_p$ and the thermal capacitance of components $C$ are considered constant, where $c_p = 4.181$ kJ/(kg.K) and $C_i = 209$ kJ/K for $i \in V \setminus \{F\}$. The reference temperatures that the system should track are the temperatures of the thermal components in the set $S = \{L, H\}$ given by $T_{H,\text{ref}} = 315$ K and $T_{L,\text{ref}} = 313$ K. The system starts with initial values $T_H(0) = 317$ K and $T_L(0) = 313$ K and the control is the mass flow rates $\dot{m}_L$ and $\dot{m}_H$ that should be designed by the controller to track the reference temperatures. Note that in this simulation we set $b = 0$ as fixed value, i.e., there is no cooling with ram air, and that would make the control effort to track the reference

These values correspond to water circulating in the FTMS loop as in [12].

| Iterations | $||T_{i+1}^L - T_{L,\text{opt}}^i||$ | $||T_{i+1}^H - T_{H,\text{opt}}^i||$ | $||T_i^J - T_{J,\text{opt}}^i||$ | $||T_{i+1}^F - T_{F,\text{opt}}^i||$ |
|------------|----------------------------------|----------------------------------|------------------------|----------------------------------|
| Iteration 1 | 28.2668                          | 18.7014                          | 2.4193                 | 137.1722                         |
| Iteration 2 | 0.3343                           | 0.7052                           | 0.0369                 | 0.0225                           |
| Iteration 3 | 0.0407                           | 0.0003                           | 0.0036                 | 0.0032                           |
| Iteration 4 | 0.0040                           | 0.0003                           | 0.0003                 | 0.0003                           |
| Iteration 5 | 0.0003                           | 0.0003                           | 0.0003                 | 0.0003                           |

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314 for $T_H$ and below 312 for $T_L$), and no cost when the temperatures are within 1 degree below the reference. The results show that the successive linearization proposed in this paper for thermal management systems outperforms the standard linearization and provides controls that better approximate the real nonlinear dynamics of the system.

Fig. 10 shows the mass of the fuel in the tank as well as the corresponding temperature during the control process.

V. CONCLUSION

In this paper, we propose applying successive linearization techniques to mitigate the effects of nonlinear dynamics in thermal management systems. It is observed that linearization around the steady-state point can produce controllers that do not capture well the nonlinear dynamics of the system due to the disturbances presented in the form of heat loads added to thermal components. Successive linearization provides promising framework to deal with the nonlinear dynamics by convexifying the nonlinearities, and iteratively converging to trajectories that ensure that the nonlinear dynamics are satisfied. The algorithm converts the original nonlinear problem into a sequence of second-order cone programs that can be solved efficiently and reliably in real-time using standard off-the-shelf solvers.

REFERENCES


Fig. 9: The temperatures at thermal components $L$ and $H$ are shown for the standard and successive linearization methods. The figure shows that successive linearization outperforms the standard one-time linearization techniques.

Fig. 10: The mass of the fuel in tank decreases due to the exit of fuel to the engine, and the temperature of the tank increases with the time for the system to track the reference temperatures.